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Novel Distance Measure Between Intuitionistic Fuzzy Sets

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Abstract: Distance measure is a basic and crucial notion of set theory. Since the intuitionistic fuzzy sets (IFSs) were put forward, distance between IFSs has been widely concerned by some researchers and many types of measures have been proposed. Most of them have counter-intuitive cases and ignore the characteristics of three parameter properties of IFSs. *The concept of fuzzy distance was introduced, which embodied the characteristics of classical distance and highlights the characteristics of hesitance index simultaneously. A new fuzzy distance measure was proposed, and the corresponding proof was included. An artificial benchmark test set of distance measure was constructed based on single-element IFSs, which was applied to compare the proposed distance measure with the widely used distance measures.* Results show that the proposed distance does not provide any counter-intuitive cases and the waver that brought from hesitancy degree can be well reflected.

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一种新的直觉模糊距离测度

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摘要：距离测度是集合理论中的一个重要基础概念。自直觉模糊集提出以来，直觉模糊距离受到了广泛的关注，近年来已有多种形式的距离测度被提出。大多数现存直觉模糊距离存在反例，且没有考虑直觉模糊集三个参数的性质。*引入了模糊距离的概念，综合体现了经典距离和犹豫指数的特点。提出了一种新的直觉模糊距离测度，并给出了相应的证明。基于单元素直觉模糊集，构建了一个人*工基准的测试集来比较本文提出的距离测度和广泛使用的距离测度。结果表明，提出的距离测度没有任何反例，并且能够很好地反映出犹豫度的波动性。

关键词：模糊距离；直觉模糊集；距离；模式识别
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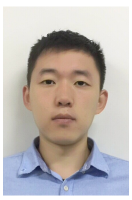
Introduction

Intuitionistic fuzzy sets (IFSs) were first put forward by Atanassov[1], which is one of the most influential generalizations of Fuzzy sets (FSs). The main feature of IFSs is that it adds a non-membership on the basis of the traditional membership, which makes it can better describe uncertainty than FSs. Although Gan and Buehrer introduced the concept of vague set later, Bustince and Burillo proved that vague sets are actually IFSs[2]. At present, IFSs has attracted much attention from researchers and has widely been applied to pattern recognition[3-5],



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medical diagnosis[6-8], decision making[9-11], approximate reasoning[12-13], logic programming[14] and cluster analysis[15-16].

As a basic and crucial notion of set theory, distance is generally applied to compare the information between two sets and calculate the degree of difference. Since the introduction of IFSs by Atanassov, distance for IFSs has quickly gained attention and several measures have been proposed. In earlier work, Szmidt and Kacprzyk[17] introduced the geometric interpretation of Intuitionistic fuzzy set (IFS) and gave four methods for computing the distances between IFSs. Later, Xu[18] put forward some weighted distances on the basis of this geometric distance model. Wang and Xin[5] held the view that the Szmidt and Kacprzyk's[17] distances have good geometric properties, but there are some limitations in the application. Therefore, some new distances have been proposed and applied in pattern recognition. On the basis of the idea of Hausdorff metric, Grzegorzewski[19] suggested some methods to measure the distances between IFSs. However, Chen[20] implied that some errors exist in Grzegorzewski's two dimensional (2D) Hausdorff based distances[19] by showing some counter-intuitive cases. Then, Yang and Chiclana[21] suggested that the three dimensional (3D) interpretation of IFSs could lead to different comparison results to the ones obtained with their 2D counterparts, and introduced several extended 3D Hausdorff based distances. In recent studies, Zhang and Yu[22] proposed two new distance measures and contrasted the advantages and disadvantages of the two approaches. Boran and Akay[23] introduced a new type of distance with two parameters and gave its relation with the similarity measure for IFSs. On the other hand, instead of establishing and improving specific distance measures, some researchers analyzed and contrasted various kinds of approaches in the past literatures. Xu and Chen[24] gave a comprehensive overview of distance and similarity measures of IFSs. Papakostas et al.[25] conducted a detailed analysis on distance and similarity measures for IFSs from the perspective of pattern recognition.

This work focuses on the distances between IFSs due to their abilities on describing uncertain information and broad application prospects. An IFS consists of a membership function, a non-membership function and a hesitance index function. These three functions can be used to describe support, opposition and neutralization of the real world respectively. Therefore, IFSs is considered as a more effective way to deal with vagueness than the conventional FSs which are characterized by only a membership. Over the past twenty years, the number of proposed distances for IFSs is constantly increasing. However, no studies have considered the information carried by hesitance index, compared to the first two functions of IFSs, with some ambiguities and uncertainties. In fact, the same value of two IFSs does not mean that their carried information is exactly the same. This characteristic has not aroused sufficient attention and most of existing distance measures[5, 18, 19, 21-23] mainly focus on particular points, which may lead to information loss and invalidation in some cases. Moreover, the axiomatic definition of distance introduced by Wang and Xin[5] may be more reasonable and targeted if the vagueness caused by the IFSs' own hesitance index is taken into account. Because the existing distance measures[5,18-19,21-23] between IFSs have these drawbacks that they cannot always get reasonable classifications when dealing with pattern recognition problems, we need to develop a new distance between IFSs to overcome the

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shortcoming.

An axiomatic definition of fuzzy distance for IFSs is introduced in this paper. A fuzzy distance measure is proposed and the corresponding proofs are provided. The results of numerical examples indicate that the proposed distance does not provide any counter-intuitive cases and the degree of difference between information carried by IFSs can be well reflected. Section 2 presents a brief introduction to the basic concepts of IFSs. In Section 3, an axiomatic definition of fuzzy distance for IFSs and the corresponding distance measure are introduced. In Section 4, a comparative analysis between the fuzzy distance and the widely used distances is conducted. In Section 5, several numerical examples for pattern recognition are presented to illustrate the usefulness of the proposed distance. The conclusion of this work is provided and the suggestions for future works are discussed in Section 6.

1 Preliminaries

Let us start a brief review of basic notions related to IFSs.

Definition 2.1 (Fuzzy sets[26]). A fuzzy set A in the universe of discourse X is an object having the form

$$A = \{< x, u_A(x) > | x \in X\} \tag{1}$$

Where： $u_A : X \to [0,1]$ is called membership function of A, $u_A(x) \in [0,1]$ represents the degree of membership of the x to A.

Definition 2.2 (Intuitionistic fuzzy sets[1]). An intuitionistic fuzzy set A in the universe of discourse X defined as follows:

$$A = \{< x, t_A(x), f_A(x) > | x \in X\} \tag{2}$$

where $t_A : X \to [0,1]$ and $f_A : X \to [0,1]$ represents membership function and non-membership function of x to A respectively. $t_A(x)$ is the lowest bound of membership degree derived from proofs of supporting x. $f_A(x)$ is the lowest bound of non-membership degree derived from proofs of opposing x, and for any $x \in X$, $0 \leqslant t_A + f_A \leqslant 1$. Obviously, when represented in the form of intuitionistic fuzzy set, fuzzy set has the form $A = \{< x, t_A(x), 1 - f_A(x) > | x \in X\}$.

The function $\pi_A(x) = 1 - t_A(x) - f_A(x)$ is called hesitance index (hesitancy degree), $\pi_A(x) \in [0,1]$ represents the degree of hesitancy of x to A. Specially, if $\pi_A(x) = 0$, $x \in X$ is known absolutely, the intuitionistic fuzzy set A degenerates into fuzzy set.

Definition 2.3 For simplicity, let $\tilde{a} = (t_a, f_a)$ be an intuitionistic fuzzy number (IFN). Then the score function of $\tilde{a}$ is defined as follows[27]:

$$s(\tilde{a}) = (t_a - f_a) \tag{3}$$

And the accuracy function of $\tilde{a}$ is defined as follows[28]:

$$h(\tilde{a}) = (t_a + f_a) \tag{4}$$

Let $\tilde{a}_1$ and $\tilde{a}_2$ be two IFNs, then Xu and Yager[29] proposed the following rules for ranking of IFNs:

(1) $s(\tilde{a}_1) < s(\tilde{a}_2)$ then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;

(2) $s(\tilde{a}_1) > s(\tilde{a}_2)$ then $\tilde{a}_2$ is smaller than $\tilde{a}_1$, denoted by $\tilde{a}_1 > \tilde{a}_2$;

(3) $s(\tilde{a}_1) = s(\tilde{a}_2)$ then

(i) $h(\tilde{a}_1) = h(\tilde{a}_2)$ then $\tilde{a}_1$ is equal to $\tilde{a}_2$, denoted by $\tilde{a}_1 = \tilde{a}_2$;

(ii) $h(\tilde{a}_1) < h(\tilde{a}_2)$ then $\tilde{a}_1$ is smaller than $\tilde{a}_2$; denoted by $\tilde{a}_1 < \tilde{a}_2$.

(iii) $h(\tilde{a}_1) > h(\tilde{a}_2)$ then $\tilde{a}_2$ is smaller than $\tilde{a}_1$; denoted by $\tilde{a}_1 > \tilde{a}_2$.

Definition 2.4 (distance measure between IFSs[5]). For any $A, B, C \in IFSs(X)$, let d be a mapping $d : IFSs(X) \times IFSs(X) \to [0,1]$. If $d(A, B)$ satisfies the following properties:

($DP1$) $0 \leqslant d(A,B) \leqslant 1$;

($DP2$) $d(A,B)=0$ if and only if A=B;

($DP3$) $d(A,B)=d(B,A)$;

($DP4$) if $A \subseteq B \subseteq C$, then $d(A,C) \geqslant d(A,B) \vee d(B,C)$.

Then $d(A,B)$ is a distance measure between IFSs A and B.

2 Fuzzy distance for Intuitionistic fuzzy sets

In this section, the axiomatic definition of fuzzy distance for IFSs is introduced firstly. Then, a new fuzzy distance measure is put forward and the related proofs are given. Furthermore, the weight form of the proposed distance is also presented.

In a very constructive work, Wang and Xin[5] introduced an axiomatic definition of distance measure between IFSs (Definition2.4). However, some essential information has been ignored in their studies and we believe that the definition may be more completed and reasonable if they have considered these information. At the beginning, Definition 2.4 just provides the value constraints when $d(A,B)$=0, the other endpoint $d(A,B)$=1 has not been discussed. Furthermore, the distance measure between IFSs could be better convinced if it satisfies the requirement of the triangle inequality. More significantly, both the membership function $t_A(x)$ and the non-membership function $f_A(x)$ represent distinct information, while the hesitance index $\pi_A(x)$ expresses the degree of hesitancy of whether x belongs to A or not, it is obvious that $\pi_A(x)$ represents uncertain information. That is to say, some ambiguities and uncertainties exist in the IFSs as long as the hesitance index is not zero. When hesitance index of A and B is not zero, we are lack of knowledge about it, so the difference between A and B should also be uncertain. On the other hand, according to common sense, it is known that the degree of uncertainty and the probability of differences are in a positive correlation. Therefore, to consider these potential differences carried by hesitance index in dealing with IFSs is of great significance. However, the existing axiomatic definition of distance between IFSs (Definition2.4) is in short of pertinence.

Considering the distance is an important measure in FSs theory and the information carried by hesitance index is characterized by uncertainty and specificity, which leads to the uncertainty of the distance calculation process, a new axiomatic definition of fuzzy distance is introduced in the following:

Definition 3.1 For any $A,B,C,D \in IFSs(X)$, let d be a mapping $d: IFSs(X) \times IFSs(X) \to [0,1]$. $d(A,B)$ is said to be a fuzzy distance between A and B if $d(A,B)$ satisfies the following properties:

($P1$) $0 \leqslant d(A,B) \leqslant 1$;

($P2$) $d(A,B)=0$ if and only if A=B and $\pi_A(x)=\pi_B(x)=0$;

($P3$) $d(A,B)=1$ if and only if both A and B are crisp sets and $A=B^C$;

($P4$) $d(A,B)=d(B,A)$;

($P5$) $d(A,C) \leqslant d(A,B)+d(B,C)$ for any $A,B,C \in IFSs(X)$;

($P6$) if $A \subseteq B \subseteq C$, then $d(A,C) \geqslant d(A,B) \vee d(B,C)$;

($P7$) if A=B and C=D, then

(i) $d(A,B) > d(C,D)$ when $\pi_A(x)+\pi_B(x) > \pi_C(x)+\pi_D(x)$;

(ii) $d(A,B) < d(C,D)$ when $\pi_A(x)+\pi_B(x) < \pi_C(x)+\pi_D(x)$;

(iii) $d(A,B) = d(C,D)$ when $\pi_A(x)+\pi_B(x) = \pi_C(x)+\pi_D(x)$;

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Let $A(x)=<x,t_A(x),f_A(x)>$, $B(x)=<x,t_B(x),f_B(x)>$ be two IFSs in the universe of discourse A=(x,1,0), denote

$$d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\theta_1(x_i)+\theta_2(x_i)+\theta_3(x_i)] \quad (5)$$

$$\theta_1(x_i)=\frac{1}{2}(|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|) \quad (6)$$

$$\theta_2(x_i)=\frac{1}{2}(\pi_A(x_i)+\pi_B(x_i)) \quad (7)$$

$$\theta_3(x_i)=\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|,\frac{|\pi_A(x_i)-\pi_B(x_i)|}{2}) \quad (8)$$

The structure of $d_L(A, B)$ is mainly related to two aspects. One is to take the differences of IFSs parameters into account, which is similar to the traditional distance. It includes differences between membership $t_A(x_i)$ and $t_B(x_i)$, non-membership $f_A(x_i)$ and $f_B(x_i)$, hesitance index $\pi_A(x_i)$ and $\pi_B(x_i)$, as well as the differences between median values of intervals $(t_A(x_i)+1-f_A(x_i))/2$ and $(t_B(x_i)+1-f_B(x_i))/2$. The second is to satisfy the requirements of fuzzy distance, taking the characteristics of IFS parameters into account. $\theta_2(x_i)$ aims to reflect the waver of uncertain information. The higher the degree of knowledge lacking of the two IFS objects, the greater the possibility of existing differences between objects information. Then we have the following theorem.

Theorem 3.1. $d_L(A,B)$ is a fuzzy distance between IFSs A and B in the universe of discourse $X=\{x_1, x_2,\ldots, x_n\}$.

Proof

P1 Let A and B be two IFSs, we can write the relational expression as following:

$$0\leqslant\theta_1(x_i)+\theta_2(x_i)=\frac{1}{2}[|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|+\pi_A(x_i)+\pi_B(x_i)]\leqslant\frac{1}{2}[|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+2-t_A(x_i)-t_B(x_i)-f_A(x_i)-f_B(x_i)]=\frac{1}{2}[2|t_A(x_i)-t_B(x_i)|-(t_A(x_i)+t_B(x_i))+2|f_A(x_i)-f_B(x_i)|-(f_A(x_i)+f_B(x_i))+2]\leqslant\frac{1}{2}[2(t_A(x_i)+t_B(x_i))-(t_A(x_i)+t_B(x_i))+2(f_A(x_i)+f_B(x_i))-(f_A(x_i)+f_B(x_i))+2]=\frac{1}{2}[(t_A(x_i)+f_A(x_i))+(t_B(x_i)+f_B(x_i))+2]\leqslant\frac{1+1+2}{2}=2 \quad (9)$$

It is known that

$$0\leqslant\theta_3(x_i)=\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|,\frac{|\pi_A(x_i)-\pi_B(x_i)|}{2})\leqslant 1 \quad (10)$$

Taking Eqs. (9) and (10) into account, it is not difficult to find that

$$0\leqslant\theta_1(x_i)+\theta_2(x_i)+\theta_3(x_i)\leqslant 3 \quad (11)$$

And therefore we have

$$0\leqslant d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\theta_1(x_i)+\theta_2(x_i)+\theta_3(x_i)]\leqslant\frac{3}{3}=1 \quad (12)$$

P2 Let A and B be two IFSs, the following relational expression can be written:

$$d_L(A,B)=0\Leftrightarrow t_A(x_i)=t_B(x_i),\ f_A(x_i)=f_B(x_i),\pi_A(x_i)+\pi_B(x_i)=0\Leftrightarrow A(x)=B(x) \text{ and } \pi_A(x)=\pi_B(x)=0$$

Thus, $d_L(A,B)$ satisfies P2 of definition3.1.

P3 Let A and B be two IFSs, taking Eqs. (9) and (10) into account, the following relational expression can be written: $d_L(A,B)=1\Leftrightarrow\theta_1(x_i)+\theta_2(x_i)=2$ and $\theta_3(x_i)=1\Leftrightarrow(t_A(x_i)+t_B(x_i)+f_A(x_i)+f_B(x_i)+2)/2=2$ and max $(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|,|\pi_A(x_i)-\pi_B(x_i)|/2)=1\Leftrightarrow A(x)=(1,0,0), B(x)=(0,1,0)$ or $A(x)=(0,1,0), B(x)=(1,0,0)$.

Thus, $d_L(A,B)$ satisfies P3 of Definition 3.1.

P4 Let A and B be two IFSs, it is known that $|t_A(x_i)-t_B(x_i)|=|t_B(x_i)-t_A(x_i)|$, $|v_A(x_i)-v_B(x_i)|=|v_B(x_i)-v_A(x_i)|$, $|\pi_A(x_i)-\pi_B(x_i)|=|\pi_B(x_i)-\pi_A(x_i)|$, $\pi_A(x_i)+\pi_B(x_i)=\pi_B(x_i)+\pi_A(x_i)$ and $|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|=|(t_B(x_i)+1-f_B(x_i))-(t_A(x_i)+1-f_A(x_i))|$.

Then, we have

$$d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|+\pi_A(x_i)+\pi_B(x_i))+\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|,\frac{|\pi_A(x_i)-\pi_B(x_i)|}{2})] \tag{13}$$

$$d_L(B,A)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_B(x_i)-t_A(x_i)|+|f_B(x_i)-f_A(x_i)|+|(t_B(x_i)+1-f_B(x_i))-(t_A(x_i)+1-f_A(x_i))|+\pi_B(x_i)+\pi_A(x_i))+\max(|t_B(x_i)-t_A(x_i)|,|f_B(x_i)-f_A(x_i)|,\frac{|\pi_B(x_i)-\pi_A(x_i)|}{2})] \tag{14}$$

Thus, $d_L(A, B)=d_L(B, A)$.

P5 Let A, B and C be three IFSs, the distances between A and B, B and C, and A and C are the following:

$$d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|+|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|+\pi_A(x_i)+\pi_B(x_i))+\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|,\frac{|\pi_A(x_i)-\pi_B(x_i)|}{2})] \tag{15}$$

$$d_L(B,C)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_B(x_i)-t_C(x_i)|+|f_B(x_i)-f_C(x_i)|+|(t_B(x_i)+1-f_B(x_i))-(t_C(x_i)+1-f_C(x_i))|+\pi_B(x_i)+\pi_C(x_i))+\max(|t_B(x_i)-t_C(x_i)|,|f_B(x_i)-f_C(x_i)|,\frac{|\pi_B(x_i)-\pi_C(x_i)|}{2})] \tag{16}$$

$$d_L(A,C)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_A(x_i)-t_C(x_i)|+|f_A(x_i)-f_C(x_i)|+|(t_A(x_i)+1-f_C(x_i))-(f_C(x_i)+1-f_A(x_i))|+\pi_A(x_i)+\pi_C(x_i))+\max(|t_A(x_i)-t_C(x_i)|,|f_A(x_i)-f_C(x_i)|,\frac{|\pi_A(x_i)-\pi_C(x_i)|}{2})] \tag{17}$$

It is obvious that

$$|t_A(x_i)-t_B(x_i)|+|t_B(x_i)-t_C(x_i)|\geqslant|t_A(x_i)-t_B(x_i)+t_B(x_i)-t_C(x_i)|=|t_A(x_i)-t_C(x_i)| \tag{18}$$

$$|f_A(x_i)-f_B(x_i)|+|f_B(x_i)-f_C(x_i)|\geqslant|f_A(x_i)-f_B(x_i)+f_B(x_i)-f_C(x_i)|=|f_A(x_i)-f_C(x_i)| \tag{19}$$

$$|(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|+|(t_B(x_i)+1-f_B(x_i))-(t_C(x_i)+1-f_C(x_i))|=|t_A(x_i)+f_B(x_i)-t_B(x_i)-f_A(x_i)|+|t_B(x_i)+f_C(x_i)-t_C(x_i)-f_B(x_i)|\geqslant|t_A(x_i)+f_B(x_i)-t_B(x_i)-f_A(x_i)+t_B(x_i)+f_C(x_i)-t_C(x_i)-f_B(x_i)|=|t_A(x_i)-f_A(x_i)+f_C(x_i)-t_C(x_i)|=|(t_A(x_i)+1-f_C(x_i))-(f_C(x_i)+1-f_A(x_i))| \tag{20}$$

$$|\pi_A(x_i)-\pi_B(x_i)|+|\pi_B(x_i)-\pi_C(x_i)|\geqslant|\pi_A(x_i)-\pi_B(x_i)+\pi_B(x_i)-\pi_C(x_i)|=|\pi_A(x_i)-\pi_C(x_i)| \tag{21}$$

$$\frac{\pi_A(x_i)+\pi_B(x_i)+\pi_B(x_i)+\pi_C(x_i)}{2}\geqslant\frac{\pi_A+\pi_C}{2} \tag{22}$$

Therefore, we have $d_L(A,B)+d_L(B,C)\geqslant d_L(A,C)$.

P6 Let A, B and C be three IFSs, if $A\subseteq B\subseteq C$, then we have $t_A(x_i)\leqslant t_B(x_i)\leqslant t_C(x_i)$, $f_A(x_i)\geqslant f_B(x_i)\geqslant f_C(x_i)$. The following equations are given out:

$$d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(t_B(x_i)-t_A(x_i)+ f_A(x_i)-f_B(x_i)+t_B(x_i)-t_A(x_i)+ f_A(x_i)-f_B(x_i)+\pi_A(x_i)+\pi_B(x_i))+ \max(t_B(x_i)-t_A(x_i), f_A(x_i)-f_B(x_i), \frac{|\pi_A(x_i)-\pi_B(x_i)|}{2})]=$$

$$\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(t_B(x_i)-t_A(x_i)+f_A(x_i)- f_B(x_i)+t_B(x_i)-t_A(x_i)+f_A(x_i)- f_B(x_i)+2-t_A(x_i)-t_B(x_i)-f_A(x_i)-f_B(x_i))+ \max(t_B(x_i)-t_A(x_i), f_A(x_i)-f_B(x_i), \frac{|t_B(x_i)-t_A(x_i)+f_B(x_i)-f_A(x_i)|}{2})]=$$

$$\frac{1}{3n}\sum_{i=1}^{n}[\frac{t_B(x_i)-3t_A(x_i)+f_A(x_i)-3f_B(x_i)}{2}+ \max(t_B(x_i)-t_A(x_i), f_A(x_i)-f_B(x_i))] \quad (23)$$

$$d_L(A,C)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(t_C(x_i)-t_A(x_i)+f_A(x_i)- f_C(x_i)+t_C(x_i)-t_A(x_i)+f_A(x_i)-f_C(x_i)+ \pi_A(x_i)+\pi_C(x_i))+\max(t_C(x_i)-t_A(x_i), f_A(x_i)- f_C(x_i), \frac{|\pi_A(x_i)-\pi_C(x_i)|}{2})]=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(t_C(x_i)- t_A(x_i)+f_A(x_i)-f_C(x_i)+t_C(x_i)-t_A(x_i)+ f_A(x_i)-f_C(x_i)+2-t_A(x_i)-t_C(x_i)-f_A(x_i)- f_C(x_i))+\max(t_C(x_i)-t_A(x_i), f_A(x_i)-f_C(x_i), \frac{|t_C(x_i)-t_A(x_i)+f_C(x_i)-f_A(x_i)|}{2})]=$$

$$\frac{1}{3n}\sum_{i=1}^{n}[\frac{t_C(x_i)-3t_A(x_i)+f_A(x_i)-3f_C(x_i)}{2}+ \max(t_C(x_i)-t_A(x_i), f_A(x_i)-f_C(x_i))] \quad (24)$$

We know that

$$t_C(x_i)-3f_C(x_i)\geqslant t_B(x_i)-3f_B(x_i),$$
$$t_C(x_i)-t_A(x_i)\geqslant t_B(x_i)-t_A(x_i),$$
$$f_A(x_i)-f_C(x_i)\geqslant f_A(x_i)-f_B(x_i) \quad (25)$$

It means that

$$d_L(A,B)\leqslant d_L(A,C) \quad (26)$$

Similarly, it is easy to prove that

$$d_L(B,C)\leqslant d_L(A,C) \quad (27)$$

Thus, $d(A,C)\geqslant d(A,B)\vee d(B,C)$.

P7 Let A, B, C and D be four IFSs, if A=B and C=D, then $t_A(x_i)=t_B(x_i)$, $f_A(x_i)=f_B(x_i)$, $\pi_A(x_i)=\pi_B(x_i)$, $t_C(x_i)=t_D(x_i)$, $f_C(x_i)=f_D(x_i)$, $\pi_C(x_i)=\pi_D(x_i)$. The following equations can be written:

$$d_L(A,B)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_A(x_i)-t_B(x_i)|+ |f_A(x_i)-f_B(x_i)|+ |(t_A(x_i)+1-f_A(x_i))-(t_B(x_i)+1-f_B(x_i))|+ \pi_A(x_i)+\pi_B(x_i))+\max(|t_A(x_i)-t_B(x_i)|, |f_A(x_i)-f_B(x_i)|, \frac{|\pi_A(x_i)-\pi_B(x_i)|}{2})]= \frac{1}{3n}\sum_{i=1}^{n}\frac{\pi_A(x_i)+\pi_B(x_i)}{2} \quad (28)$$

$$d_L(C,D)=\frac{1}{3n}\sum_{i=1}^{n}[\frac{1}{2}(|t_C(x_i)-t_D(x_i)|+ |f_C(x_i)-f_D(x_i)|+ |(t_C(x_i)+1-f_C(x_i))-(t_D(x_i)+1-f_D(x_i))|+ \pi_C(x_i)+\pi_D(x_i))+\max(|t_C(x_i)-t_D(x_i)|, |f_C(x_i)-f_D(x_i)|, \frac{|\pi_C(x_i)-\pi_D(x_i)|}{2})]= \frac{1}{3n}\sum_{i=1}^{n}\frac{\pi_C(x_i)+\pi_D(x_i)}{2} \quad (29)$$

Then, we have

(i) if $\pi_A(x_i)+\pi_B(x_i)>\pi_C(x_i)+\pi_D(x_i)$ then $d(A,B)>d(C,D)$

(ii) if $\pi_A(x_i)+\pi_B(x_i)<\pi_C(x_i)+\pi_D(x_i)$ then $d(A,B)<d(C,D)$

(iii) if $\pi_A(x_i)+\pi_B(x_i)=\pi_C(x_i)+\pi_D(x_i)$ then $d(A,B)=d(C,D)$

Therefore, $d_L(A,B)$ satisfies P7 of definition 3.1.

That is to say $d_L(A,B)$ is a fuzzy distance between IFSs A and B since it satisfies ($P1$)-($P7$).

3 Performance evaluation

In this section, several widely used distances between IFSs are firstly recalled. Then, an artificial benchmark test set of distance based on single-element IFSs is constructed, and the test set is applied to compare the proposed distance measure to the widely used distance measures.

E. Szmidt and J. Kacprzyk[17] proposed four distances between IFSs using the well-known Hamming distance, Euclidean distance and their normalized counterparts as follows:

$$d_H(A,B)=\frac{1}{2}\sum_{i=1}^{n}[|t_A(x_i)-t_B(x_i)|+ |f_A(x_i)-f_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|] \quad (30)$$

$$d_{nH}(A,B)=\frac{1}{2n}\sum_{i=1}^{n}[|t_A(x_i)-t_B(x_i)|+ |f_A(x_i)-f_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|] \quad (31)$$

$$d_E(A,B)= \sqrt{\frac{1}{2}\sum_{i=1}^{n}[(t_A(x_i)-t_B(x_i))^2+(f_A(x_i)-\rightarrow} \overline{\leftarrow\frac{1}{2}\sum_{i=1}^{n} f_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2]} \quad (32)$$

$$d_{nE}(A,B)= \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[(t_A(x_i)-t_B(x_i))^2+(f_A(x_i)-\rightarrow} \leftarrow\frac{1}{2n}\sum_{i=1}^{n} f_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2] \quad (33)$$

These distances pay attention to the D-value of $t_A(x_i)$ and $t_B(x_i)$, $f_A(x_i)$ and $f_B(x_i)$, $\pi_A(x_i)$ and $\pi_B(x_i)$.

Wang and Xin[5] noticed some drawbacks of E. Szmidt and J. Kacprzyk's distances[17] and put forward some new approaches.

$$d_1(A,B)= \frac{1}{n}\sum_{i=1}^{n}[\frac{|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|}{4}+ \frac{\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|)}{2}] \quad (34)$$

$$d_{1w}(A,B)= \sum_{i=1}^{n}w_i[\frac{|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|}{4}+ \frac{\max(|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|)}{2}]/\sum_{i=1}^{n}w_i \quad (35)$$

$$d_2^p(A,B)= \frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\frac{|t_A(x_i)-t_B(x_i)|+|f_A(x_i)-f_B(x_i)|}{2})^p} \quad (36)$$

$d_{1w}(A,B)$ is a form of weighted distance to $d_1(A,B)$, where $w_i=1/n,\ i\in\{1,2,...,n\}$.

On the basis of the Hausdorff metric, Grzegorzewski[19] put forward some approaches for gauging distances between IFSs, and these suggested distances are also generalizations of the well-known Hamming distance, Euclidean distance and their normalized counterparts.

$$d_h(A,B)= \sum_{i=1}^{n}\max\{|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|\} \quad (37)$$

$$l_h(A,B)= \frac{1}{n}\sum_{i=1}^{n}\max\{|t_A(x_i)-t_B(x_i)|,|f_A(x_i)-f_B(x_i)|\} \quad (38)$$

$$e_h(A,B)= \sqrt{\sum_{i=1}^{n}\max\{(t_A(x_i)-t_B(x_i))^2,(f_A(x_i)-f_B(x_i))^2\}} \quad (39)$$

$$q_h(A,B)= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\{(t_A(x_i)-t_B(x_i))^2,(f_A(x_i)-f_B(x_i))^2\}} \quad (40)$$

Yang and Chiclana[21] suggested that the 3D interpretation of IFSs could provide different contradistinction results to the ones obtained with their 2D counterparts[16], and introduced several extended 3D Hausdorff based distances.

$$d_{eh}(A,B)=\sum_{i=1}^{n}\max\{|t_A(x_i)-t_B(x_i)|, |f_A(x_i)-f_B(x_i)|,|\pi_A(x_i)-\pi_B(x_i)|\} \quad (41)$$

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$$l_{eh}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\{|t_A(x_i)-t_B(x_i)|, |f_A(x_i)-f_B(x_i)|, |\pi_A(x_i)-\pi_B(x_i)|\} \quad (42)$$

$$e_{eh}(A,B)=\sqrt{\sum_{i=1}^{n}\max\{(t_A(x_i)-t_B(x_i))^2,(f_A(x_i)-f_B(x_i))^2,(\pi_A(x_i)-\pi_B(x_i))^2\}} \quad (43)$$

$$q_{eh}(A,B)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\{(t_A(x_i)-t_B(x_i))^2,(f_A(x_i)-f_B(x_i))^2,(\pi_A(x_i)-\pi_B(x_i))^2\}} \quad (44)$$

Zhang and Yu [22] put forward two new distance measures and discussed their advantages and disadvantages. Assume that both IFSs A and B contain only one element, the first distance measure is as follows:

Suppose that $t_A(x)\leqslant t_B(x)$, and sort the sequence $(t_A(x),1-f_A(x),t_B(x),1-f_B(x))$ in descending order: (a,b,c,d).

$$d_z^1(A,B)=\begin{cases}(U-I)/2, & 1-f_A\geqslant t_B\\ (U+I)/2, & 1-f_A<t_B\end{cases} \quad (45)$$

where $U=a-d$ and $I=b-c$.

The main steps of the second distance measure are listed below:

IFSs A and B are respectively transformed into symmetric triangular fuzzy numbers $\tilde{A}=(t_A(x), m_A(x),1-f_A(x))$ and $\tilde{B}=(t_B(x),m_B(x),1-f_B(x))$, where $m_A(x)=(t_A(x)+1-f_A(x))/2$ and $m_B=(t_B(x)+1-f_B(x))/2$. Suppose that $m_A(x)\leqslant m_B(x)$. Let $t_{\tilde{A}}(x)$ be the membership function of symmetric triangular fuzzy number $\tilde{A}$:

$$t_{\tilde{A}}(x)=\begin{cases}(x-t_A)/(m_A-u_A), & t_A(x)\leqslant x\leqslant m_A(x)\\ (1-f_A-x)/(1-f_A-m_A), & m_A(x)\leqslant x\leqslant 1-f_A(x)\\ 0, & \text{otherwise}\end{cases} \quad (46)$$

$$d(A,B)=d(\tilde{A},\tilde{B})=U-I \quad (47)$$

where: $I=\int_0^1\min(t_{\tilde{A}}(x),t_{\tilde{B}}(x))\mathrm{d}x$ (48)

and $U=$

$$\int_0^{m_A}\max(t_{\tilde{A}}(x),t_{\tilde{B}}(x))dx+|m_B(x)-m_A(x)|+\int_{m_B}^1\max(t_{\tilde{A}}(x),t_{\tilde{B}}(x))dx \quad (49)$$

Boran and Akay [23] introduced a new distance for IFSs with two parameters.

$$d_f^d(A,B)=\sqrt[p]{\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\{|t(t_A(x_i)-t_B(x_i))-(f_A(x_i)-f_B(x_i))|^p+|t(f_A(x_i)-f_B(x_i))-(t_A(x_i)-t_B(x_i))|^p} \quad (50)$$

where t=2,3,4,… and p=1,2,3,…

When a new distance measure is proposed, it's always accompanied with explanations of overcoming counter-intuitive cases of other methods. Instead of multi-element sets, the counter-intuitive cases are usually illustrated in the form of single-element IFSs. There are three main reasons in the following：To begin with, the single-element IFSs are fundamental to the multi-element IFSs, and the counter-intuitive cases of multi-element IFSs can be constructed according to that of one-element counter-intuitive cases. In the second place, the numerical analysis results on the basis of single-element can explain drawbacks of some distance measures accurately and clearly, while the multi-element IFSs tend to conceal disadvantages because it is of complex and overlap. In the third place, the comparisons based on single-element IFSs are simple, compact and easy for people to understand. So in order to prove the effectiveness of the proposed distance, we construct an artificial benchmark test set consisting of single-element IFSs, and use it to assess the proposed distance and some widely applied distances. As shown in Tab. 1, these single-element IFSs are mainly counter-intuitive cases proposed by previous literature about distances between IFSs (both general academic papers and review articles). Although these cases cannot represent all counter- intuitive situations, they are typical and representative. Specifically, the following types are included:

Type Ⅰ: The distance between A and B calculated by some measures are equal to 1 when $\{A=(x,1,0),B=(x,0,0)\}$ and $\{A=(x,0,0),B=(x,0.5,0.5)\}$, which seems unreasonable because it does not obey distance measure property condition P3.

Type Ⅱ: The distance between A and B are equal to that between C and D calculated by some measures when $A=(x,0.3,0.3)$, $B=(x,0.4,0.4)$, $C=(x,0.3,0.4)$ and $D=(x,0.4,0.3)$, which indicates that there are not sufficient abilities to distinguish positive difference from negative difference.

TypeIII: The distance between A and B calculated by some measures is not equal to zero when $\{A=(x,0.6,0.4),B=(x,0.6,0.4)\}$, which does not obey distance measure property condition P2.

TypeIV: Another type of counter-intuitive case takes place when $A=(x,1,0)$, $B=(x,0,0)$ and $C=(x,0.5,0.5)$. In this case, the distance between A and B is equal to that between B and C, which are actually not equivalent to each other.

TypeV: Another type of counter-intuitive example can be given when $A=(x,0.4,0.2)$, $B=(x,0.5,0.3)$ and $C=(x,0.5,0.2)$. In this case, the distance between A and B calculated by some measures is equal to or greater than that between A and C, which does not seem to be reasonable since IFSs A, B and C are ordered as $C>B>A$ according to the score function and accuracy function given in Definition 2.3, indicating that the distance between A and B is smaller than that between A and C. Similar counter-intuitive case exists when $A=(x,0.5,0.3)$, $B=(x,0.5,0.2)$ and $C=(x,0.4,0.2)$. The distance between A and B is equal to that between B and C, which are indeed not equal to each other according to Definition 2.3.

TypeVI: Another particular counter-intuitive case could be considered for future studies when $A=(x,0.1,0.2)$ and $B=(x,0.1,0.2)$. Although with great uncertainty in this situation (high hesitance index denotes that $x \in X$ is barely known), the distance between A and B calculated by some measures is equal to zero, which does not seem to be reasonable. As a mathematical tool, IFSs can describe the uncertain information greatly, because it adds a hesitant index to describe the state of “both this and that”. The ultimate goal of distance measure is to measure the difference of information carried by IFSs, rather than the difference of IFSs numerical value itself. Therefore, the distance measure should have its own target. In other words, IFSs have the advantage of being able to consider waver (degree of hesitancy). However, these measures have not considered the waver in the process of comparing information carried by two IFSs. We indeed can’t confirm that there is no difference between the information carried by IFSs A and B, because the hesitance index includes some uncertain information and the proportion of support and opposition is not sure.

Tab. 2 provides a comprehensive comparison of the distance measures for IFSs with counter-intuitive cases. It is apparent that the axiomatic definition of fuzzy distance (P3) is not met by d_{nH}, d_{nE}, l_h, l_{eh}, because the distances calculated by these measures are equal to 1 when $\{A=(x,1,0),B=(x,0,0)\}$. Similarly, the axiomatic definition of the fuzzy distance is also not

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satisfied by d_{nH}, l_{eh} when $\{A=(x,0,0), B=(x,0.5,0.5)\}$ (type1). The distance measures l_h, d_1, $d_2^P, p=1$ and d_z^1 claim that the distance of the 6th test IFSs and the 7th test IFSs show the same difference of 0.1, which seems to be incorrect(type 2). Distance measures d_{nH}, l_{eh}, $d_2^P, p=1$, and d_z^1 indicate that the distance of the 2nd test IFSs and the 3rd test IFSs are identical, which is not correct obviously (type 4). Another type of counter-intuitive case can be given in which the distances calculation of the 4th test IFSs and 10th test IFSs are compared, the distance of 4th test IFSs is equal to or greater than the distance of the 10th test IFSs when d_{nH}, d_{nE}, l_h, l_{eh}, d_1, $d_2^P, p=1$, d_z^1 are used. The outcomes are actually not adequate (type 5). The similar counter-intuitive case occurs for d_{nH}, d_{nE}, l_h, l_{eh}, d_1, $d_2^P, p=1$, d_z^1, $d_f^d, t=2, p=1$ when the 5th test IFSs and the 10th test IFSs are compared, which the distance of the 5th test IFSs is not actually equivalent to that of the 10th test IFSs regarding to Definition 2.3 (type 5). What is more, the distance measure $d_z^2(A,B)$ is ineffective when dealing with the 1st test IFSs, the 2nd test IFSs, the 3rd test IFSs, the 8th test IFSs and the 11th test IFSs, because m_A minus u_A is equal to zero. In a similar way, $d_z^2(A,B)$ is not effective in the 3rd test IFSs since m_B minus u_B is equal to 0. Furthermore, all of these existing distances claim that the distance of the 9th test IFSs is equal to 0, which does not seem to be reasonable (type 6).

Based on analysis in Tab. 2, it is deduced that the existing distance measures with their own measuring focus can meet all or most of properties condition of distance measure between IFSs, however, most distance measures show many counter-intuitive cases and may fail to distinguish IFSs accurately in some practical applications. Besides, the proposed distance measure is the only one that has no aforementioned counter-intuitive cases as illustrated in Tab. 2. Furthermore, the proposed distance conforms to all the property requirements of the fuzzy distance and the potential difference brought by hesitance index is considered.

Tab. 1　Test intuitionistic fuzzy sets (IFSs)

Test IFSs	1	2	3	4	5	6	7	8	9	10	11
$A=(t_A, f_A)$	(1,0)	(1,0)	(0,0)	(0.4,0.2)	(0.5,0.3)	(0.3,0.3)	(0.3,0.4)	(0.6,0.4)	(0.1,0.2)	(0.4,0.2)	(1,0)
$B=(t_B, f_B)$	(0,1)	(0,0)	(0.5,0.5)	(0.5,0.3)	(0.5,0.2)	(0.4,0.4)	(0.4,0.3)	(0.6,0.4)	(0.1,0.2)	(0.5,0.2)	(0.5,0.5)

Tab. 2　Comparison of distance measures (Counter-intuitive cases are in bold italic type)

Measure	Test IFSs										
	1	2	3	4	5	6	7	8	9	10	11
$d_L(A,B)$	1.000 0	0.833 3	0.500 0	0.166 7	0.150 0	0.166 7	0.200 0	0	0.233 3	0.183 3	0.500 0
$d_{nH}(A,B)$	1.000 0	***1.000 0***	***1.000 0***	***0.200 0***	***0.100 0***	0.200 0	0.100 0	0	**0**	***0.100 0***	0.500 0
$d_{nE}(A,B)$	1.000 0	***1.000 0***	0.866 0	***0.173 2***	***0.100 0***	0.173 2	0.100 0	0	**0**	***0.100 0***	0.500 0
$l_h(A,B)$	1.000 0	***1.000 0***	0.500 0	***0.100 0***	***0.100 0***	***0.100 0***	***0.100 0***	0	**0**	***0.100 0***	0.500 0
$l_{eh}(A,B)$	1.000 0	***1.000 0***	***1.000 0***	***0.200 0***	***0.100 0***	0.200 0	0.100 0	0	**0**	***0.100 0***	0.500 0
$d_1(A,B)$	1.000 0	***0.750 0***	0.500 0	***0.100 0***	***0.075 0***	***0.100 0***	***0.100 0***	0	**0**	***0.075 0***	0.500 0
$d_2^P(A,B)$	1.000 0	***0.500 0***	**0.500 0**	***0.100 0***	***0.050 0***	***0.100 0***	***0.100 0***	0	**0**	***0.050 0***	0.500 0
$d_z^1(A,B)$	1.000 0	***0.500 0***	***0.500 0***	***0.100 0***	***0.050 0***	***0.100 0***	***0.100 0***	0	**0**	***0.050 0***	0.500 0
$d_z^2(A,B)$	***NaN***	***NaN***	***NaN***	0.090 0	0.095 0	0.100 0	0.183 3	***NaN***	**0**	0.096 5	***NaN***
$d_f^d(A,B)$	1.000 0	0.500 0	0.166 7	0.033 3	***0.050 0***	0.033 3	0.100 0	0	**0**	***0.050 0***	0.500 0

4 Conclusion

Various types of distance measures between IFSs were proposed over the past several years. But many of these distances have provided counter-intuitive results and only consider the differences between the numerical values of the IFSs parameters. In this paper, we introduced an axiomatic definition of the fuzzy distance for IFSs, which not only considered the value difference between two IFSs, but the nature characteristics of information carried by IFSs. And then we proposed a new distance measure and proved that it is a fuzzy distance by satisfying all the property requirements of Definition 3.1. Furthermore, we presented some comparisons between the existing distances and the proposed distance. The numerical results demonstrated that the proposed distance does not provide any counter-intuitive cases and it is more consistent with actual circumstances than existing measures in the difference calculation with uncertainties. This work indicated that the same value of two IFSs cannot imply that there is no difference in their carried information. Therefore, the significance and the specificity of the hesitancy degree should be considered when computing distance for IFSs. As far as future directions are concerned, we hope that our proposed distance measure can be applied in some fields such as pattern recognition, decision making and approximate reasoning. It would also be interesting to extend the research idea into other extensions of FSs such as interval-valued intuitionistic fuzzy sets (IVIFSs)[30] and hesitant fuzzy sets (HFSs)[31].

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